

ON LOCALIZATION OF SCHRÖDINGER MEANS

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ABSTRACT. Localization properties for Schrödinger means are studied in dimension higher than one.

1. INTRODUCTION

Let f belong to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ where $n \geq 1$. We define the Fourier transform \hat{f} by setting

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

For $f \in \mathcal{S}(\mathbb{R}^n)$ we also set

$$(1.1) \quad S_t f(x) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} e^{it|\xi|^2} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

If we set $u(x, t) = S_t f(x)/(2\pi)^n$, then $u(x, 0) = f(x)$ and u satisfies the Schrödinger equation $i\partial u/\partial t = \Delta u$.

It is well-known that $e^{i|\xi|^2}$ has the Fourier transform $K(x) = ce^{-i|x|^2/4}$, where c denotes a constant, and $e^{it|\xi|^2}$ has the Fourier transform

$$K_t(x) = \frac{1}{t^{n/2}} K\left(\frac{x}{t^{1/2}}\right), \quad x \in \mathbb{R}^n, \quad t > 0.$$

One has $S_t f(x) = K_t \star f(x)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$, and we set

$$(1.2) \quad S_t f(x) = K_t \star f(x), \quad t > 0,$$

for $f \in L^1(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}^n)$ we define $S_t f$ by formula (1.1).

We introduce Sobolev spaces $H_s = H_s(\mathbb{R}^n)$ by setting

$$H_s = \{f \in \mathcal{S}' : \|f\|_{H_s} < \infty\}, \quad s \in \mathbb{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$

In the case $n = 1$ it is well-known (see Carleson [3] and Dahlberg and Kenig [4]) that

$$\lim_{t \rightarrow 0} \frac{1}{2\pi} S_t f(x) = f(x)$$

almost everywhere if $f \in H_{1/4}$. Also it is known that $H_{1/4}$ cannot be replaced by H_s if $s < 1/4$. In the case $n \geq 2$ Sjölin [6] and Vega [9] proved independently that

$$\lim_{t \rightarrow 0} \frac{1}{(2\pi)^n} S_t f(x) = f(x)$$

almost everywhere if $f \in H_s$ and $s > 1/2$. This result was improved by Bourgain [1], who proved that $f \in H_s(\mathbb{R}^n)$, $s > 1/2 - 1/4n$, is sufficient for convergence almost everywhere.

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In the case $n = 2$, Du, Guth, and Li [5] have recently proved that the condition $s > 1/3$ is sufficient. On the other hand Bourgain [2] has proved that $s \geq n/2(n+1)$ is necessary for convergence for all $n \geq 2$.

We shall here study localization of Schrödinger means and shall first state a result on localization everywhere (see Sjölin [7]).

Theorem A. *Assume $n \geq 1$. If $f \in H_{n/2}(\mathbb{R}^n)$ and f has compact support then*

$$\lim_{t \rightarrow 0} S_t f(x) = 0$$

for every $x \in \mathbb{R}^n \setminus (\text{supp } f)$.

It is also proved in [7] that this result is sharp in the sense that $H_{n/2}$ cannot be replaced by H_s with $s < n/2$.

We say that one has localization almost everywhere for functions in H_s if for every $f \in H_s$ one has

$$\lim_{t \rightarrow 0} S_t f(x) = 0$$

almost everywhere in $\mathbb{R}^n \setminus (\text{supp } f)$.

In the case $n = 1$ Sjölin and Soria proved that there is no localization almost everywhere for functions in H_s if $s < 1/4$ (see Sjölin [8]). In fact they proved that there exist two disjoint compact intervals I and J in \mathbb{R} and a function f which belongs to H_s for every $s < 1/4$, with the properties that $\text{supp } f \subset I$ and for every $x \in J$ one does not have

$$\lim_{t \rightarrow 0} S_t f(x) = 0.$$

In the case $n \geq 2$ Sjölin and Soria also proved that one does not have localization almost everywhere for functions in $H_s(\mathbb{R}^n)$ if $s < 1/4$.

We shall here improve this result and prove that there is no localization almost everywhere for functions in $H_s(\mathbb{R}^n)$ if $n \geq 2$ and $s < n/2(n+1)$. In fact we shall prove the following theorem.

Theorem 1.1. *If $n \geq 2$ and $s < n/2(n+1)$ there exist a function f in $H_s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and a set F with positive Lebesgue measure such that $F \subset \mathbb{R}^n \setminus (\text{supp } f)$ and for every $x \in F$ one does not have $\lim_{t \rightarrow 0} S_t f(x) = 0$.*

To prove this result we shall combine the method in [8] with an estimate of Bourgain [2].

If A and B are numbers we write $A \lesssim B$ if there is a positive constant C such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$ we write $A \sim B$.

We introduce the inverse Fourier transform by setting

$$\check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

for $f \in L^1(\mathbb{R}^n)$.

Also $B(x; r)$ denotes a ball with center x and radius r .

2. PROOF OF THE THEOREM

We start by taking v_1 such that $0 < v_1 < 1$ and set $v_k = \varepsilon_k v_{k-1}^\mu$ for $k = 2, 3, 4, \dots$, where $\varepsilon_k = 2^{-k}$ and $\mu = \max(n, 2 + n/4)$. Then one has $v_k < 2^{-k}$ for $k \geq 2$ and $(v_k)_1^\infty$ is a decreasing sequence tending to zero.

We then choose $g \in \mathcal{S}(\mathbb{R})$ such that $\text{supp } \check{g} \subset (-1, 1)$, $\check{g}(x_1) = 1$ for $|x_1| < 1/2$, and set $f_v(x_1) = e^{-ix_1/v^2} \check{g}(x_1/v)$, $0 < v < 1$, $x_1 \in \mathbb{R}$.

The functions f_v were used in Sjölin [8] to study the localization problem in the case $n = 1$. Also let $\Phi \in \mathcal{S}(\mathbb{R}^{n-1})$ have $\text{supp } \hat{\Phi} \subset B(0; 1)$ and $\Phi(0) = 1$. We then take $R = 1/v^2$ and set

$$G_v(x') = R^{-(n-1)/4} \Phi(x') \prod_{j=2}^n \left(\sum_{R/2D < l_j < R/D} e^{iDl_j x_j} \right), \quad 0 < v < 1,$$

where $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, $l = (l_2, \dots, l_n) \in \mathbb{Z}^{n-1}$, and $D = R^{(n+2)/2(n+1)}$. We may also assume that $\Phi(x') = \psi(x_2) \dots \psi(x_n)$ for some $\psi \in \mathcal{S}(\mathbb{R})$.

We then set

$$h_v(x) = h_v(x_1, x') = f_v(x_1) G_v(x'), \quad 0 < v < 1.$$

In [2] Bourgain studies functions similar to h_v . However, in [2] our function \check{g} is replaced by a function φ with the property that $\hat{\varphi}$ has compact support. In our argument it will be important that \check{g} has compact support so that

$$\text{supp } f_v \subset (-v, v).$$

We then observe that

$$S_t h_v(x_1, x') = S_t f_v(x_1) S_t G_v(x').$$

It is proved in Bourgain [2], p.394, that if one assumes $|x| < c$ and $|t| < c/R$ and sets

$$(2.1) \quad t = \frac{x_1}{2R} + \tau$$

with $|\tau| < R^{-3/2}/10$, then

$$(2.2) \quad |S_t f_v(x_1)| \gtrsim |\check{g}(R^{1/2}x_1 - 2tR^{3/2})| = |\check{g}(2\tau R^{3/2})| \geq c_0.$$

We then take $v = v_k$ for $k = 1, 2, 3, \dots$, and apply an estimate in [2], p.395, namely that there exists a set $E_k \subset B(0; 1)$ such that for every $x \in E_k$ there exists $t = t_k(x)$ such that

$$|S_{t_k(x)} G_{v_k}(x')| \gtrsim R^{-(n-1)/4} \prod_{j=2}^n \left| \sum_{l_j} e^{iDl_j x_j} e^{iD^2 l_j^2 t} \right| \geq c_0.$$

Also one has $mE_k \geq c_1 > 0$, where m denotes Lebesgue measure.

We then choose $\delta > 0$ so small that if $F_k = E_k \cap \{x; |x_1| > \delta/2\}$ then one has $mF_k \geq c_1/2 = c_2$ for $k = 1, 2, 3, \dots$. We may assume that $\delta < 1$.

We then set $F = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} F_j \right)$ so that F is the set of all x which belong to infinitely

many F_k . Since the sets $\bigcup_{j=k}^{\infty} F_j$ decrease as k increases, one obtains $mF \geq c_0$.

It also follows from (2.2) that $|S_{t_k(x)} f_{v_k}(x)| \geq c_0$ for $x \in F_k$. From (2.1) one also concludes that

$$(2.3) \quad |t_k(x)| \sim \frac{1}{R_k} = v_k^2$$

where $R_k = 1/v_k^2$.

We now choose K so large that $v_K < \delta/4$ and set

$$h(x) = \sum_{k=K}^{\infty} h_{v_k}(x) = \sum_{k=K}^{\infty} f_{v_k}(x_1) G_{v_k}(x').$$

One has $F \subset \mathbb{R}^n \setminus (\text{supp } h)$ since $\text{supp } h \subset \{x; |x_1| < \delta/4\}$.

If $x = (x_1, x') \in F$ there exists a sequence $(k_j)_{j=1}^{\infty}$ such that $x \in F_{k_j}$ and

$$|S_{t_{k_j}(x)} G_{v_{k_j}}(x')| \geq c_0 \quad \text{and} \quad |S_{t_{k_j}(x)} f_{v_{k_j}}(x_1)| \geq c_0.$$

We shall prove that

$$|S_{t_{k_j}(x)}h(x)| \geq c_0$$

for j large, and also that $h \in H_s \cap L^1$ for $s < n/2(n+1)$. It follows that one does not have localization almost everywhere in H_s if $s < n/2(n+1)$.

We shall first estimate $\|h_v\|_{H_s}$. We begin by studying f_v and G_v . According to Sjölin [8], p. 143, one has

$$\widehat{f}_v(\xi_1) = vg(v\xi_1 + 1/v) = R^{-1/2}g(R^{-1/2}\xi_1 + R^{1/2})$$

and for $|\xi_1| \geq AR$ we have

$$R^{-1/2}|\xi_1| \geq R^{1/2}A,$$

where A is a large constant. It follows that $|\xi_1| \geq AR$ implies

$$|R^{-1/2}\xi_1 + R^{1/2}| \geq BR^{-1/2}|\xi_1|$$

and

$$|\xi_1| \geq |\xi_1|^{1/2}A_1R^{1/2}.$$

Hence

$$|R^{-1/2}\xi_1 + R^{1/2}| \geq B_1|\xi_1|^{1/2}$$

and

$$|\widehat{f}_v(\xi_1)| \leq C|\xi|^{-N} \quad \text{for } |\xi_1| \geq AR,$$

where N is large. It follows that

$$(2.4) \quad \int_{|\xi_1| \geq AR} |\widehat{f}_v(\xi_1)|^2 (1 + \xi_1^2)^s d\xi_1 \leq CR^{-N}$$

and it is also easy to see that

$$(2.5) \quad \|f_v\|_2 \sim v^{1/2}.$$

We have

$$\widehat{G}_v(\xi') = R^{-(n-1)/4} \sum_l \widehat{\Phi}(\xi' - Dl)$$

and it follows that $\text{supp } \widehat{G}_v \subset \{\xi'; |\xi'| \sim R\}$ and

$$|\widehat{G}_v(\xi')|^2 = R^{-(n-1)/2} \sum_l |\widehat{\Phi}(\xi' - Dl)|^2.$$

Integrating we obtain

$$\|\widehat{G}_v\|_2^2 = R^{-(n-1)/2} \sum_l \|\widehat{\Phi}\|_2^2 \sim R^{-(n-1)/2} \left(\frac{R}{D}\right)^{n-1}.$$

We have $D = R^{(n+2)/2(n+1)}$ and $R/D = R^{n/2(n+1)}$ and hence

$$\|G_v\|_2^2 \sim R^{-(n-1)/2} R^{n(n-1)/2(n+1)} = R^{-(n-1)/2(n+1)}$$

and

$$(2.6) \quad \|G_v\|_2 \sim R^{-(n-1)/4(n+1)}.$$

For $s > 0$ one obtains

$$\begin{aligned} \|h_v\|_{H_s}^2 &\sim \int |\widehat{h}_v(\xi)|^2 |\xi|^{2s} d\xi = \int_{|\xi'| \sim R} |\widehat{f}_v(\xi_1)|^2 |\widehat{G}_v(\xi')|^2 |\xi|^{2s} d\xi \lesssim \\ &\iint_{\substack{|\xi'| \sim R \\ |\xi_1| \leq AR}} |\widehat{f}_v(\xi_1)|^2 |\widehat{G}_v(\xi')|^2 R^{2s} d\xi_1 d\xi' + \iint_{\substack{|\xi'| \sim R \\ |\xi_1| \geq AR}} |\widehat{f}_v(\xi_1)|^2 |\widehat{G}_v(\xi')|^2 |\xi_1|^{2s} d\xi_1 d\xi' = I_1 + I_2. \end{aligned}$$

It follows that

$$\begin{aligned} I_1 &\lesssim R^{2s} \left(\int_{|\xi_1| \leq AR} |\widehat{f}_v(\xi_1)|^2 d\xi_1 \right) \left(\int |\widehat{G}_v(\xi')|^2 d\xi' \right) \lesssim \\ &R^{2s} \|f_v\|_2^2 \|G_v\|_2^2 \lesssim R^{2s} R^{-1/2} R^{-(n-1)/2(n+1)} = R^{2s-n/(n+1)}. \end{aligned}$$

From (2.4) and (2.6) one also obtains

$$I_2 \lesssim R^{-N}$$

and hence

$$I_1 + I_2 \lesssim R^{2s-n/(n+1)}.$$

It follows that

$$\|h_v\|_{H_s} \lesssim R^{s-n/2(n+1)} = v^{2(n/2(n+1)-s)}.$$

Since $v_k < \varepsilon_k$ and

$$\|h_k\|_{H_s} \leq \sum_K \|h_{v_k}\|_{H_s} \lesssim \sum_K v_k^{2(n/2(n+1)-s)} < \infty,$$

it follows that $h \in H_s$ if $s < n/2(n+1)$.

We also need some estimates for $S_t f_v$ and $S_t G_v$. In Sjölin [8] (see Lemmas 3 and 4) it is proved that

$$(2.7) \quad |S_t f_v(x_1)| \lesssim \frac{v}{|t|^{1/2}}$$

and

$$(2.8) \quad |S_t f_v(x_1)| \lesssim \frac{|t|}{v^4}$$

for $0 < v < \delta/4$, $|x_1| \geq \delta/2$, and $0 < |t| < 1$. Actually it is assumed in [8] that $t > 0$ but the same proofs work for $t < 0$.

We also have the following estimates for $S_t G_v$.

Lemma 2.1. *For $0 < v < \delta/4$, $0 < |t| < 1$, and $x' \in \mathbb{R}^{n-1}$ one has*

$$(2.9) \quad |S_t G_v(x')| \lesssim v^{(n-1)/2} (\log 1/v)^{n-1} |t|^{-(n-1)/2}$$

and

$$(2.10) \quad |S_t G_v(x')| \lesssim v^{-(n-1)^2/2(n+1)}.$$

Proof. Choose the integer p so that $|4p - R/D| \leq 4$. Then one has

$$\sum_{R/2D < l < R/D} e^{iDlx_j} = \sum_{2p}^{4p} e^{iDlx_j} + O(1)$$

and

$$\left| \sum_{2p}^{4p} e^{iDlx_j} \right| = \left| \sum_{-p}^p e^{iD(l+3p)x_j} \right| = \left| \sum_{-p}^p e^{ilDx_j} \right| = D_p(Dx_j),$$

where D_p denotes the Dirichlet kernel. Setting $y = Dx_j$ one obtains

$$\int_a^{a+1/D} |D_p(Dx_j)| dx_j = \int_{Da}^{Da+1} |D_p(y)| dy \frac{1}{D} \lesssim \frac{1}{D} \log p \sim \frac{1}{D} \log R$$

for every $a \in \mathbb{R}$. It follows that

$$\int_a^{a+1} |D_p(Dx_j)| dx_j \lesssim \log R$$

for every $a \in \mathbb{R}$.

Letting Q denote the cube $[0, 1]^{n-1}$ we obtain

$$\begin{aligned} \|G_v\|_1 &= \int_{\mathbb{R}^{n-1}} |G_v(x')| dx' = \sum_{m \in \mathbb{Z}^{n-1}} \int_{m+Q} |G_v(x')| dx' \lesssim \\ &R^{-(n-1)/4} \sum_m \frac{1}{1 + |m|^N} \int_{m+Q} \left(\prod_{j=2}^n \left| \sum_{l_j} e^{il_j Dx_j} \right| \right) dx' \lesssim \\ &R^{-(n-1)/4} \sum_m \frac{1}{1 + |m|^N} \prod_{j=2}^n \left(\int_{m_j}^{m_j+1} \left| \sum_{l_j} e^{il_j Dx_j} \right| dx_j \right) \lesssim \\ &R^{-(n-1)/4} \sum_m \frac{1}{1 + |m|^N} (\log R)^{n-1} \end{aligned}$$

and hence

$$\|G_v\|_1 \lesssim R^{-(n-1)/4} (\log R)^{n-1} \sim v^{(n-1)/2} (\log 1/v)^{n-1}.$$

We have

$$S_t G_v(x') = \int K_t(x' - y') G_v(y') dy'$$

where

$$|K_t(y')| \lesssim |t|^{-(n-1)/2}$$

and it follows that

$$|S_t G_v(x')| \lesssim |t|^{-(n-1)/2} \|G_v\|_1$$

and hence we obtain (2.9).

We also have

$$S_t G_v(x') = \int_{\mathbb{R}^{n-1}} e^{i\xi' \cdot x'} e^{it|\xi'|^2} \widehat{G}_v(\xi') d\xi' = R^{-(n-1)/4} \sum_l \int_{\mathbb{R}^{n-1}} e^{i\xi' \cdot x'} e^{it|\xi'|^2} \widehat{\Phi}(\xi' - Dl) d\xi'$$

and we get

$$|S_t G_v(x')| \lesssim R^{-(n-1)/4} \sum_l \|\widehat{\Phi}\|_1 \lesssim R^{-(n-1)/4} (R/D)^{n-1} = R^{(n-1)^2/4(n+1)} = v^{-(n-1)^2/2(n+1)}$$

and the proof of Lemma 2.1 is complete. \square

Multiplying (2.7) and (2.9) one obtains

$$(2.11) \quad |S_t h_v(x)| \lesssim v^{(n+1)/2} (\log 1/v)^{n-1} |t|^{-n/2} \lesssim \frac{v}{|t|^{n/2}} = \frac{v}{|t|^\gamma}$$

and combining (2.8) and (2.10) one gets

$$(2.12) \quad |S_t h_v(x)| \lesssim \frac{|t|}{v^{4+(n-1)^2/2(n+1)}} \lesssim \frac{|t|}{v^{4+n/2}} = \frac{|t|}{v^\beta}$$

where $\gamma = n/2$, $\beta = 4 + n/2$, $0 < v < \delta/4$, $0 < |t| < 1$, and $|x_1| > \delta/2$.

We remark that it also follows from the above estimates that $h \in L^1(\mathbb{R}^n)$. In fact it is easy to see that $\|f_v\|_1 \sim v$ and we have proved that

$$\|G_v\|_1 \lesssim v^{(n-1)/2} (\log 1/v)^{n-1} \lesssim v^{1/4}$$

and hence $\|h_v\|_1 \lesssim v^{5/4}$. It follows that

$$\|h\|_1 \leq \sum_K^\infty \|h_{v_k}\|_1 < \infty.$$

We shall now finish the proof of the following result.

Theorem 2.2. *For $x \in F$ one has*

$$|S_{t_{k_j}(x)} h(x)| \geq c_0 > 0$$

for j large. It follows that there is no localization almost everywhere of Schrödinger means for functions in $H_s(\mathbb{R}^n)$ if $s < n/2(n+1)$.

Proof. We have

$$|S_{t_{k_j}(x)} h(x)| \geq |S_{t_{k_j}(x)} h_{v_{k_j}}(x)| - \left| \sum_{i \neq k_j} S_{t_{k_j}(x)} h_{v_i}(x) \right|.$$

The first term on the right hand side is larger than a positive number c_0 and it suffices to prove that the second term is small. For simplicity we write k instead of k_j in the following formulas.

We have $0 < v_i \leq v_{i-1}/2$ and $0 < v_i \leq 2^{-i}$ and it follows that

$$(2.13) \quad \sum_{i=k+1}^\infty v_i \leq 2v_{k+1}$$

and one also has

$$(2.14) \quad \sum_{i=K}^{k-1} \frac{1}{v_i^\beta} \lesssim \frac{1}{v_{k-1}^\beta}$$

since $1/v_{i-1} \leq 1/2v_i$ implies $1/v_{i-1}^\beta \leq \frac{1}{2^\beta} \frac{1}{v_i^\beta}$.

For $i \geq k+1$ the inequality (2.11) and the formula (2.3) give

$$|S_{t_k(x)} h_{v_i}(x)| \lesssim \frac{v_i}{(v_k^2)^\gamma} = \frac{v_i}{v_k^{2\gamma}}$$

and invoking (2.13) we obtain

$$\sum_{i=k+1}^\infty |S_{t_k(x)} h_{v_i}(x)| \lesssim \frac{v_{k+1}}{v_k^{2\gamma}} \lesssim \varepsilon_{k+1},$$

since $\mu \geq 2\gamma$ and $v_{k+1} = \varepsilon_{k+1} v_k^\mu \leq \varepsilon_{k+1} v_k^{2\gamma}$. For $K \leq i \leq k-1$ the inequality (2.12) gives

$$|S_{t_k(x)} h_{v_i}(x)| \lesssim \frac{|t_k(x)|}{v_i^\beta} \lesssim \frac{v_k^2}{v_i^\beta}$$

and invoking (2.14) one obtains

$$(2.15) \quad \sum_{i=K}^{k-1} |S_{t_k(x)} h_{v_i}(x)| \lesssim v_k^2 \sum_K^{k-1} \frac{1}{v_i^\beta} \lesssim \frac{v_k^2}{v_{k-1}^\beta}.$$

Since $\mu \geq \beta/2$ we get $v_k \leq \varepsilon_k v_{k-1}^{\beta/2}$ and $v_k^2 \leq \varepsilon_k^2 v_{k-1}^\beta$. Hence the sum in (2.15) is majorized by $C\varepsilon_k^2$. Since $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$ the proof of the theorem is complete. \square

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